Fourier and Laplace Transforms

Symbolical computes the Fourier and Laplace transforms of a function by symbolic evaluation (→ operator): type the expression, then Ctrl+. and the keyword of the transform, followed by two variables, the one of the function and the one of its transform.

Fourier transform

The keyword fourier, t, ν gives F(ν) = ∫ f(t)·exp(−2πi·ν·t) dt over all the values of t, ν being the frequency (in Hz when t is in seconds). The Gaussian is its own transform:

$$e^{-\pi t^2} \text{ fourier}, t, \nu \to e^{-\pi \nu^2}$$

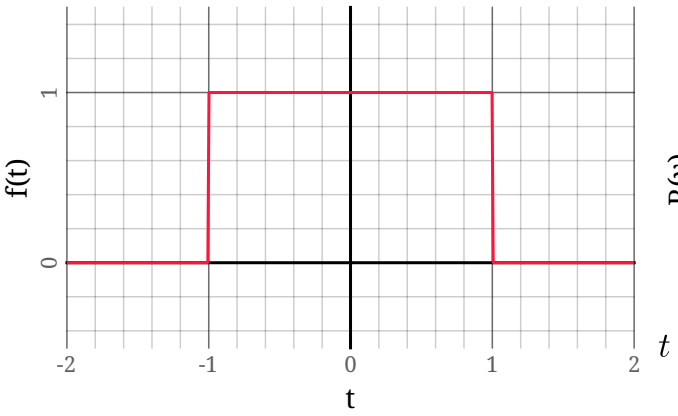
A rectangular pulse of width 2, written with the Heaviside step Φ, has a cardinal sine for transform. The result is assigned to the function R(ν):

$$R(\nu) := \Phi(1 - |t|) \text{ fourier}, t, \nu \to \frac{\sin(2\,\pi\,\nu)}{\pi\,\nu}$$

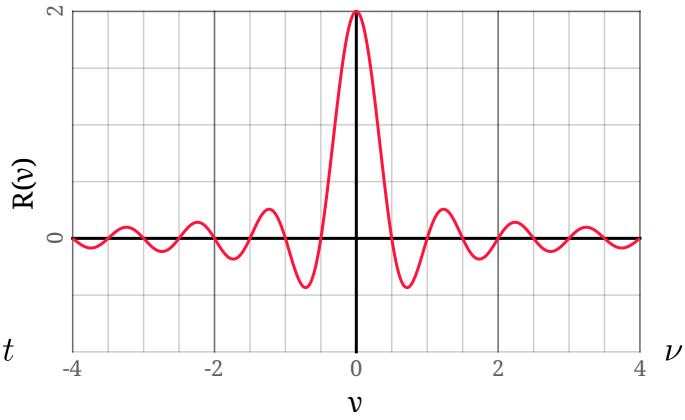
Check at ν = 0.3 by a numerical integral:

$$R(0.3) = 1.009 \qquad \int_{-1}^{1} e^{-2\,\pi\,i\cdot 0.3\,t}\,\mathrm{d}t = 1.009$$

Rectangular pulse
$\Phi(1 - |t|)$



Its Fourier transform
$R(\nu)$

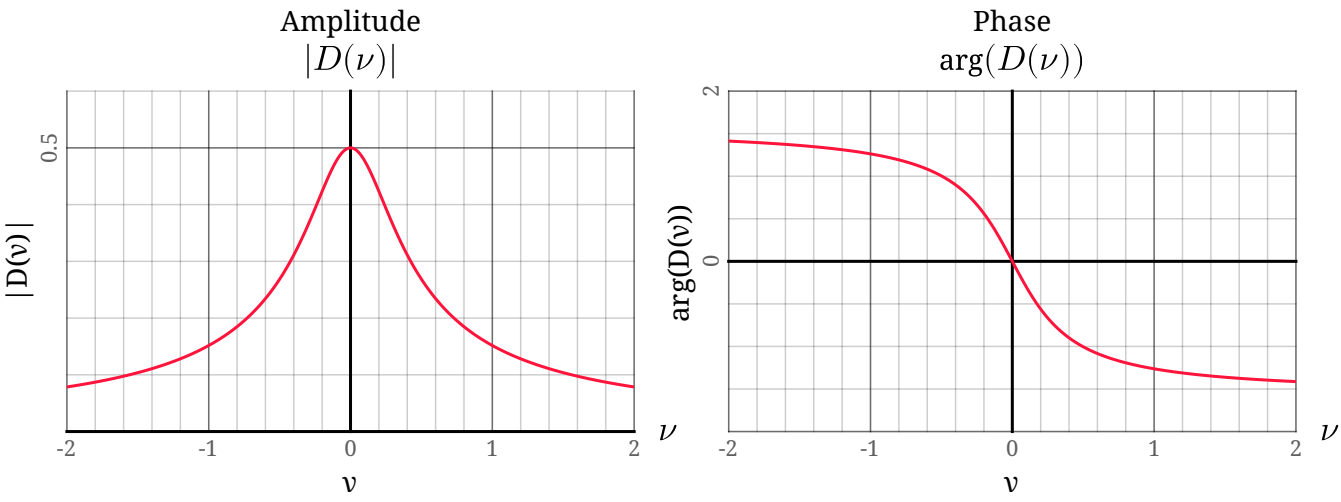


The inverse transform is given by invfourier, ν, t. A two-sided exponential and back:

$$e^{-|t|} \text{ fourier}, t, \nu \to \frac{2}{4\,\pi^2\,\nu^2 + 1} \qquad \frac{2}{4\,\pi^2\,\nu^2 + 1} \text{ invfourier}, \nu, t \to e^{-|t|}$$

The transform of a function that is not even is complex: here, an exponential decay starting at t = 0. Its modulus is the amplitude spectrum, its argument the phase:

$$D(\nu) := \Phi(t)\,e^{-2\,t} \text{ fourier}, t, \nu \to \frac{1}{2\,(i\,\pi\,\nu + 1)}$$



Some transforms are made of distributions, as the Dirac deltas of the transform of cos(t): Symbolical then answers "No symbolic form found".

Laplace transform

The keyword laplace, t, s gives L(s) = $\int$ f(t)·exp(−s·t) dt for t from 0 to ∞. The names that are not defined, such as ω or a, stay symbols:

$$\sin(\omega\, t)\ \text{laplace}, t, \text{s} \to \frac{\omega}{\text{s}^2 + \omega^2} \qquad \cos(\omega\, t)\ \text{laplace}, t, \text{s} \to \frac{\text{s}}{\text{s}^2 + \omega^2}$$

$$e^{-a\, t}\ \text{laplace}, t, \text{s} \to \frac{1}{a + \text{s}} \qquad t\, e^{-a\, t}\ \text{laplace}, t, \text{s} \to \frac{1}{(a + \text{s})^2}$$

The inverse transform invlaplace, s, t gives the function for t ≥ 0. A delay e^(−s) gives a delayed step, a first-order system the response to a unit step, and complex poles a damped oscillation:

$$\frac{e^{-\text{s}}}{\text{s}}\ \text{invlaplace}, \text{s}, t \to \Phi(t - 1) \qquad \frac{1}{\text{s}\,(1 + \tau\,\text{s})}\ \text{invlaplace}, \text{s}, t \to 1 - e^{-\frac{t}{\tau}}$$

$$\frac{\text{s}}{\text{s}^2 + 2\,\text{s} + 5}\ \text{invlaplace}, \text{s}, t \to e^{-t}\ \cos(2\, t) - \frac{e^{-t}\ \sin(2\, t)}{2}$$

Solving a differential equation

Let y″ + 3·y′ + 2·y = 1, with y(0) = 0 and y′(0) = 0. As L[y′] = s·Y − y(0) and L[y″] = s²·Y − s·y(0) − y′(0), the Laplace transform turns it into an equation in Y, solved symbolically:

$$Y(s) := s^2\, X + 3\, s\, X + 2\, X = \frac{1}{s}\ \text{solve}, X \to \frac{1}{\text{s}\,(\text{s}^2 + 3\,\text{s} + 2)}$$

In partial fractions, each term is the transform of an exponential:

$$Y(\text{s})\ \text{partfrac}, \text{s} \to \frac{1}{2\,(\text{s} + 2)} - \frac{1}{\text{s} + 1} + \frac{1}{2\,\text{s}}$$

and the inverse transform gives the solution:

$$y(t) := Y(\mathrm{s}) \text{ invlaplace, s, } t \rightarrow \frac{1}{2} - e^{-t} + \frac{e^{-2\,t}}{2}$$

The same equation solved numerically by odesolve:

Solve block

$z(0) = 0$

$z'(0) = 0$

$z''(t) + 3 \cdot z'(t) + 2\,z(t) = 1$

$z := \text{odesolve}(t, 6)$

Both solutions agree:

$y(2) = 0.374$ $\quad$ $z(2) = 0.374$

